Final Report for ONR Grant N00014-90-J-1206
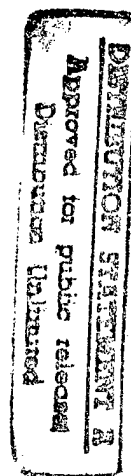
§1.   Introduction.

This final report covers the time period from December 1, 1989 to November 30, 1995 of ONR Grant N00014-90-J-1206. It is organized as follows. In this section I will give statistical information covering this period. The next five sections progress in the major areas of my research is summarized and the main accomplishments are highlighted. The last section provides references to my papers and other papers discussed in the summary. It is assumed that the reader is familiar with my original proposals [64] and [65].

The papers [27] - [43], [45] - [49], [51], [53], [56], and [57] were published during this period. The papers [44], [50], [52], [55], and [58] were accepted for publication. The papers [54] and [60] were submitted for publication. The papers [29], [39], [41] - [44], [53], and [60] are about on-line coloring. The papers [37], [45], and [54] concern the closely related subject of game chromatic number. The papers [33], [35] - [36], [38], [46], [50], and [57] deal with Gyárfás' graph coloring conjecture on which many of the on-line coloring results rest. The papers [28], [30], [34], [47], [49], [52], and [58] concern the dimension of ordered sets, while [27], [31], [32], and [40] deal with other concepts related to ordered sets. Finally [48], [51], [55], and [56] concern a conjecture of Pósa about square cycles in dense graphs.

The following students, who all received some support from the grant, have all earned doctorate degrees from Arizona State University: Stephen Penrice, Jun Qin, Katalin Kolossa, Juan Quintana, and Yingxing Zhu.

§2.   Gyárfás' Conjecture.

Much of my work on on-line coloring is founded on the study of off-line coloring, and in particular, progress on the following conjecture of Gyárfás. For two graphs G and H, let Forb(G, H) denote the class of graphs that induce neither G nor H. Let $\chi(G)$ denote

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If you have questions, please contact Joe Wessels, Sponsored Projects Officer, at (602) 965-1427.

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Enclosures

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denote the chromatic number of G and $\omega(G)$ denote the clique number of G. Gyárfás made the following conjecture.

Conjecture 2.1. (Gyárfás [68]) For any tree T and clique K, there exists a bound b such that for every graph $G \in Forb(T, K)$, $\chi(G) \leq b$.

Gyárfás [69] gave easy arguments to show that the conjecture is true if T is a path or a star. Gyárfás, Szemerédi, and Tuza [70] showed that the conjecture is true if T is a radius two tree and K is a triangle. My student Steve Penrice and I made the following major breakthrough.

Theorem 2.2. (Kierstead and Penrice[35]) For every radius two tree T and every clique K, there exists a bound b such that for every graph $G \in Forb(T, K)$, $\chi(G) \leq b$.

This result depended on the notion of template first used in [33]. Over a series of papers [36], [38], [46], [50], and [57] with various coauthors from the list Penrice, Rödl, Trotter, and Zhu, I sharpened this technique to deal with trees of larger radius. With my student Yingxian Zhu I proved the following generalization of Theorem 2.3, which is exciting because the same type of preliminary result [33] in the radius two case led to the proof of Theorem 2.2.

Theorem 2.3. (Kierstead and Zhu [57]) Let T be the radius three tree such that every vertex adjacent to the root has degree two. Then Conjecture 2.1 holds for T.

An important related question has been raised by Sauer. A class of graphs Γ is said to be *vertex Ramsey* if for every $G \in \Gamma$ there exists $H \in \Gamma$ such that if the vertices of H are partitioned into two sets inducing subgraphs H_1 and H_2, then either H_1 or H_2 contains an induced copy of G.

Conjecture 2.4. (Sauer [71]) The class Forb(T, K) is not Ramsey for all trees T and cliques K.

It is easy to show that Conjecture 2.1 implies Conjecture 2.5. What makes Conjecture 2.5 especially interesting from my perspective, is that the same template techniques that have been useful for dealing with special cases of Conjecture 2.1 are also useful for more general cases of Conjecture 2.5. Moreover, these more general cases of Conjecture 2.5 have led to the development of more sophisticated versions of the template technique. I believe that it is likely that we will be able to use these more sophisticated versions to make further progress on Conjecture 2.1. Here are some examples. A *spider with toes* is a tree such that the nonleaf vertices form the subdivision of a star. Let $T(k,r)$ be the tree such that the distance between the root and any leaf is r and every nonleaf has k sons. The following two theorems give the only examples of trees for which Conjecture 2.5 is known to hold, but Conjecture 2.1 is not known to hold. In some sense, a spider with toes is a thin tree, while $T(k,r)$ is a full fat tree when k is large. (Of course, $T(1,r)$ is a spider with toes.)

Theorem 2.6. (Kierstead [50]) For every spider with toes T and clique K, Forb(T, K) is not vertex Ramsey.

Theorem 2.7 (Kierstead and Zhu [57]) For every positive integer r and sufficiently large integer k and every clique K, Forb(T(k,r), K) is not vertex Ramsey.

I believe that we are now closing in on the techniques necessary to solve the following problem in particular and possibly all of Conjecture 2.1.

Problem 2.8. Prove Conjecture 2.1 for the case of radius three trees.

§3. On-line Coloring

On-line coloring has been the major focus of this project and I have proved four especially important results. The first of these concerns the application of an on-line coloring algorithm to the off-line problem Dynamic Storage Allocation (DSA). DSA is an NP-Complete off-line problem. In the late sixties an off-line approximation algorithm, based on First-Fit coloring of interval graphs, was suggested for DSA. The performance ratio of this algorithm was twice the competitive ratio of First-Fit. However the best upper bound on the competitive ratio of First-Fit was $O(\omega)$. During the seventies this was improved to $O(\log \omega)$, but the question of whether it was constant remained open until 1986 when I [25] provided a upper bound of 40 for First-Fit coloring of interval graphs, and thus 80 for the performance ratio for DSA. I have now significantly improved this result. I observed that with some care First-Fit could be replaced by a better on-line algorithm for coloring interval graphs. This allowed me to prove the following theorem.

Theorem 3.1. (Kierstead [29]) There exists a polynomial time approximation algorithm for DSA that has a performance ration of 6.

The intuition behind this algorithm is the following. DSA resembles both Bin Packing and Interval Graph Coloring. There exist orderings for both problems, under which First-Fit performs well, but they are not the same. Thus we order the data so that First-Fit performs well for Bin Packing and use an on-line algorithm for interval graph coloring that is guaranteed to perform reasonably well under any input sequence.

My other on-line coloring results concern on-line coloring per se. The next result is based on Theorem 2.2, but is much harder. It identifies classes of graphs with the property that their on-line chromatic number can be bounded in terms of their clique number.

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Theorem 3.2. (Kierstead, Penrice, and Trotter [42]) For every radius two tree T and every clique K, there exists an on-line algorithm A and a bound b such that for every graph $G \in \text{Forb}(T, K)$, $\chi_A(G) \leq b$.

In particular, this answers an old question of Schmerl, Problem 7 of my 1989 ONR proposal [64], who asked whether the on-line chromatic number of co-comparability graphs could be bounded in terms of their clique size. The answer is yes since co-comparability graphs do not induce the radius two tree obtained from a star with three leaves by subdividing each edge.

For any integer $k \geq 2$ the on-line chromatic number of the class of graphs with chromatic number k is unbounded as the number of vertices n increases. Thus in general we must must express bounds in terms of n. Lovász, Saks, and Trotter went to considerable effort to prove the upper bound of $O\left(\dfrac{\log^{(2k-3)} n}{\log^{(2k-4)} n}\right)$, which is barely sublinear. Very recently, I have done much better. I proved the following Theorem, which answers Problem 1 of my 1992 ONR proposal [65].

Theorem 3.3. (Kierstead [60]) There exists an on-line algorithm A that will properly color any k-colorable on-line graph on n vertices with $n^{1-1/k!}$ colors.

My student Katie Kolossa and I had previously obtained much better upper bounds for perfect graphs. Moreover our lower bounds for perfect graphs match the best lower bounds for graphs in general. This answers Problem 2 of my 1992 ONR proposal [65].

Theorem 3.4. (Kierstead and Kolossa [53]) There exists an on-line algorithm A that will properly color any k-colorable on-line perfect graph on n vertices with $n^{10k/\log\log n}$ colors. Moreover for every fixed positive integer k and every on-line algorithm A, there exists an on-line k-colorable perfect graph G such that A uses at least $\dfrac{1}{2}\left(\dfrac{\log_3 n}{2k}\right)^{k-1}$ colors on G.

§4. Game Chromatic Number

In [37] Faigle, Kern, Trotter and I developed the notion of game chromatic number of a graph. A game is played between two players A and B on a graph G. The players play by taking turns coloring the vertices of a graph with colors from a fixed set of n colors so that no two adjacent vertices receive the same color. If at some point in this process one of the players cannot play legally, then B wins; otherwise, when the graph is completely colored A wins. The game chromatic number $\chi_g(G)$ of G is the least n such that A has a winning strategy when the game is played on the graph G with n colors. Our main result was that the game chromatic number of a tree is at most four. We also introduced the problem of whether there exists a constant upper bound on the game chromatic number of planar graphs.

This question was answered by Trotter and me. We introduced the notion of a k-admissible ordering of a graph and proved the following two theorems, which together show that every planar graph has game chromatic number at most 33. We denote the least k such that a graph G has a k-admissible ordering by a(G).

Theorem 4.1. (Kierstead and Trotter [45]) For every graph G, $\chi_g(G) \leq \chi(G)a(G) + 1$

Theorem 4.2. (Kierstead and Trotter [45]) For every planar graph G, $a(G) \leq 8$. Moreover there exist planar graphs with $a(G) = 8$.

Tuza and I generalized the bound on game chromatic number of trees by phrasing it in terms of the treewidth $\tau(G)$ of an arbitrary graph G. This answers Problem 10 of my 1992 ONR proposal [65].

Theorem 4.3. (Kierstead and Tuza [54]) For every graph G, $\tau(G) \leq 6\tau(G)-2$.

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§5. Dimension of Ordered Sets

An important problem in the theory of ordered sets is to bound the dimension of various ordered sets. I have been able too answer several questions that had been open for many years. One such problem concerns the relation between the dimension of an ordered set and the maximum degree Δ of its comparability graph. In a beautiful paper Füdedi and Kahn proved the following inequality, where $g(\Delta)$ is the maximum dimension of an ordered set whose comparability graph has maximum degree Δ.

Theorem 5.1. (Füredi and Kahn [67]) For all Δ, $g(\Delta) = O(\Delta \log^2 \Delta)$.

The only problem with this result was that nobody new of any ordered sets with dimension greater than $\Delta+1$. Erdös, Trotter, and I used random bipartite ordered sets to provide such an example and answer Problem 12 of my 1989 ONR proposal. Let $\Omega(n,p)$ be the probability space consisting of bipartite ordered sets with n maximal elements and n minimal elements where the event that a given maximal element is greater than a given minimal element is independently distributed with probability p.

Theorem 5.2. (Erdös, Kierstead, and Trotter [34]) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$n^{-1+\varepsilon} \le p \le \frac{1}{\log n},$$

then for almost all $P \in \Omega(n,p)$, $\dim(P) > \delta\Delta(P) \log |P| > \delta\Delta(P) \log \Delta(P)$.

We also proved:

Theorem 5.3. (Erdös, Kierstead, and Trotter [34]) There exist constants c_1 and c_2 such that for almost all labeled ordered sets P, $\frac{n}{4}\left(1 - \frac{c_1}{\log n}\right) < \dim(P) < \frac{n}{4}\left(1 + \frac{c_2}{\log n}\right)$.

Furedi and Kahn obtained the bound of Theorem 5.1 in terms of the dimension of a certain subset of the boolean lattice. Let B_n denote the Boolean lattice obtained by ordering the

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power set of $\{1, 2,..., n\}$ by inclusion. Let $B_n(j, k)$ be the restriction of B_n to subsets of size j and k and $\dim(j, k; n) = \dim(B_n(j,k))$. Furedi and Kahn actually proved the following two lemmas.

Lemma 5.4. (Füredi and Kahn [67]) For all ordered sets P with $\Delta(P) = \Delta$,
$$\dim(P) = O(\frac{\Delta}{\log \Delta} \dim(1, \log \Delta; \Delta \log \Delta)).$$

Lemma 5.5. (Füredi and Kahn [67]) For all positive integers $k < n$, $\dim(1, k; n) \leq (k+1)^2 \log n$. In particular, $\dim(1, \log n; n) = O(\log^3 n)$.

The $\log \Delta$ gap between the example of Theorem 5.2 and Theorem 5.1 made the question of determining $\dim(1, \log n; n)$ important. I proved:

Theorem 5.6. (Kierstead [52]) $\dfrac{\log^3 n}{64 \log \log n} < \dim(1, \log n; n).$

In [49] Brightwell, Kostochka, Trotter, and I studied the function $\dim(i, j; n)$ for $1 < i < j$.

§6. Square Cycles in Dense Graphs

An important problem in graph theory is to show that all dense graphs contain certain spanning subgraphs. For example Dirac's famous theorem says that every graph on n vertices with minimum degree at least $n/2$ has a hamiltonian cycle. In 1963 Pósa made the following conjecture.

Conjecture 6.1. (Pósa [66]) Every graph on n vertices with minimum degree at least $\frac{2}{3}n$ contains the square of a hamiltonian cycle, i.e., a hamiltonian cycle together with every 2-chord.

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In a series of papers with Genghua Fan, I have come very close to proving this conjecture. First we showed that it was asymptotically correct in [48]. Then we proved the following theorem.

Theorem 6.2. (Fan and Kierstead [51]) Every graph on n vertices with minimum degree at least $(2n-1)/3$ contains the square of a hamiltonian path.

This already implies the important result of Aigner and Brandt that every graph on n vertices with minimal degree at least $(2n-1)/3$ contains every graph on n vertices with maximum degree 2. We also proved:

Theorem 6.3. (Fan and Kierstead [56]) Let G be a graph on n vertices with minimum degree $\frac{2}{3}n$. If G does not contain a square hamiltonian cycle then:

(1) G does not contain a square cycle of length greater than $\frac{2}{3}n$; and

(2)   The vertices of G can be partitioned into two square cycles.

With my student Juan Quintana [55], I showed that the conjecture is true if G has a maximal 4-clique.

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